GLOBAL WELL-POSEDNESS FOR A MODIFIED 2D DISSIPATIVE QUASI-GEOSTROPHIC EQUATION WITH INITIAL DATA IN THE CRITICAL SOBOLEV SPACE H^1

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Abstract. In this paper, we consider the following modified quasi-geostrophic equations

(MQG) $$\partial_t\theta + \Lambda^{\alpha}\theta + u\vec{\nabla}\theta = 0, \quad u = \Lambda^{\alpha-1}\mathcal{R}^{\perp}(\theta)$$

wher $\alpha \in]0,1[$ is a fixed parameter. This equation was recently introduced by P. Constantin, G. Iyer and J. Wu in [4] as a modification of the classical quasi-geostrophic equation. In this paper, we prove that for any initial data θ_* in the Sobolev space $H^1(\mathbb{R}^2)$, the equation (MQG) has a global and smooth solution θ in $C\left(\mathbb{R}^+, H^1\left(\mathbb{R}^2\right)\right)$.

1. Introduction and statement of results

In this paper, we are concerned with the modified 2D dissipative quasi-geostrophic equation:

(MQG) $$\begin{cases} \partial_t\theta + \Lambda^{\alpha}\theta + u\vec{\nabla}\theta = 0 \\ u = \Lambda^{\alpha-1}\mathcal{R}^{\perp}(\theta) \\ \theta|_{t=0} = \theta_*. \end{cases}$$

Here, $\alpha \in]0,1[$ is a fixed real number, the unknown $\theta = \theta(t,x)$ is a real-valued function defined on $\mathbb{R}^+ \times \mathbb{R}^2$, θ_* is a given initial data, $\mathcal{R}^{\perp}$ is the operator defined via Riesz transforms by;

$$\mathcal{R}^{\perp}(\theta) = \left(-\mathcal{R}_2\theta, \mathcal{R}_1\theta\right),$$

and Λ^{γ} is the non-local operator defined through the Fourier transform by:

$$\widehat{\Lambda^{\gamma} f}\ (\xi) = |\xi|^{\gamma}\, \hat{f}\ (\xi).$$

The equation (MQG) was recently introduced in [4] by P. Constantin, G. Iyer and J. Wu as a modification of the 2D dissipative quasi-geostrophic equation

(QG) $$\begin{cases} \partial_t\theta + \Lambda^{\alpha}\theta + u\vec{\nabla}\theta = 0 \\ u = \mathcal{R}^{\perp}(\theta) \\ \theta|_{t=0} = \theta_*. \end{cases}$$

In [4], the autors proved that if the initial data θ_* belongs to $L^2(\mathbb{R}^2)$ then the equation (MQG) has a global solution $\theta \in C^{\infty}\left(]0,+\infty[\times\mathbb{R}^2\right)$.

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Very recently, C. Miao and L. Xue in [10] have proved that for any initial data $\theta_* \in H^m(\mathbb{R}^2)$, with $m \in \mathbb{N}$ and $m > 2$, there exists a unique global solution $\theta \in C(\mathbb{R}^+, H^m(\mathbb{R}^2))$ to the equation (MQG). Moreover,the solution θ satisfies the following regularity property:

$$\forall \gamma \geq 0,\ t^\gamma \theta \in L^\infty(\mathbb{R}^+, H^{m+\gamma\sigma}(\mathbb{R}^2)).$$

One of the main properties of the equation (MQG) is the following scaling invariance property: If θ is a solution to (MQG) with initial data θ_* then, for all $\lambda > 0$, the function $\theta_\lambda \equiv \theta(\lambda^\alpha t, \lambda x)$ is a solution to (MQG) with initial data $\theta_*^\lambda \equiv \theta_*(\lambda x)$. This leads us to introduce the notion of critical space: a functional space X is called a critical space for the equation (MQG) if for all $f \in X$ and $\lambda > 0$, we have

$$\|f(\lambda .)\|_X = \|f\|_X .$$

For instance, the homogenous Sobolev's space $\dot{H}^\sigma(\mathbb{R}^2)$ is a critical space if and only if $\sigma = 1$. Therefore, following the classical approach of Fujita-Kato [8], it is natural to ask if the equation (MQG) is well-posed if the initial data θ_* belongs to the critical space $\dot{H}^1(\mathbb{R}^2)$. In this paper, we give a slightly weaker result. In fact, we prove the global well-posedness of the smooth solution to the equation (MQG) when the initial data is in the inhomogeneous Sobolev's space $H^1(\mathbb{R}^2)$. Precisely, our result states as follows

Theorem 1.1. *Let $\theta_* \in H^\sigma(\mathbb{R}^2)$ with $\sigma \geq 1$. Then there is a unique solution θ in*

$$C^1\left(\mathbb{R}^+, L^2\left(\mathbb{R}^2\right)\right) \cap C\left(\mathbb{R}^+, H^\sigma\left(\mathbb{R}^2\right)\right) \cap L^2_{loc}\left(\mathbb{R}^+, H^{\sigma+\frac{\alpha}{2}}\left(\mathbb{R}^2\right)\right)$$

to the equation (MQG). Moreover, for all $\sigma' \geq \sigma$ we have

$$\theta \in C^\infty\left(\mathbb{R}^+_*, H^{\sigma'}\left(\mathbb{R}^2\right)\right).$$

In particular, $\theta \in C^\infty\left(\mathbb{R}^+_* \times \mathbb{R}^2\right)$.

The proof of this theorem relies essentially on the following two propositions. The first one is a local well-posedness result.

Proposition 1.1. *Let $\theta_* \in H^\sigma(\mathbb{R}^2)$ with $\sigma \geq 1$. Then the equation (MQG) has a unique maximal solution θ belonging to the space*

$$C^1\left([0, T^*[, L^2\left(\mathbb{R}^2\right)\right) \cap C\left([0, T^*[, H^\sigma\left(\mathbb{R}^2\right)\right) \cap L^2_{loc}\left([0, T^*[, H^{\sigma+\frac{\alpha}{2}}\left(\mathbb{R}^2\right)\right).$$

Moreover, the time T^* is bounded from below by

$$\sup\{T > 0 :\ \mathcal{K}(\theta_*, T) \geq \varepsilon_\sigma\}$$

where $\varepsilon_\sigma > 0$ is a constant depending only on σ, and

$$\mathcal{K}(\theta_*, T) = \left\| \left(\left[\frac{1 - e^{-2\nu 2^{\alpha q} T}}{2\nu} \right]^{1/2} 2^{\sigma q} \left\|\Delta_q \theta_*\right\|_2 \right)_q \right\|_{l^2(\mathbb{Z})}$$

where $\nu > 0$ is an absolute constant and $(\Delta_q)_q$ denotes the family of the Littlewood-Paley operators (for the definition, see the section 2).

The second result concerns the propagation of the initial regularity.

Proposition 1.2. *Let $\theta_* \in H^1(\mathbb{R}^2)$ and let θ be a solution to the equation (MQG) belonging to the space*

$$C^1\left([0,T], L^2\left(\mathbb{R}^2\right)\right) \cap C\left([0,T], H^1\left(\mathbb{R}^2\right)\right) \cap L^2\left([0,T], H^{1+\frac{\alpha}{2}}\left(\mathbb{R}^2\right)\right).$$

If there exists $t_0 \in [0,T[$ and $\sigma \geq 1$ such that $\theta(t_0)$ belongs to $H^\sigma(\mathbb{R}^2)$, then the solution θ belongs to the space

$$C\left([t_0,T], H^\sigma\left(\mathbb{R}^2\right)\right) \cap L^2\left([t_0,T], H^{\sigma+\frac{\alpha}{2}}\left(\mathbb{R}^2\right)\right).$$

2. Notations and preliminaries

In this preparatory section, we shall introduce some functionals spaces and prove some elementary lemmas that will be used in the proof of Theorem 1.1.

2.1. **Notations.**

(1) Throughout this paper, we will denotes various constants by C. In particular, $C = C_{*,*,\dots}$ denotes constants depending only on the quantities appearing in the index.
(2) Let A and B be two reals functions. The notation $A \lesssim B$ means that there exists a constant $c > 0$ such that $A \leq cB$. We write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.
(3) For $p \in [1,\infty]$, we denote by L^p the Lebesgue space $L^p(\mathbb{R}^2)$ endowed with the usual norm $\|.\|_p$.
(4) Let $T > 0$, $r \in [1,\infty]$ and X be a Banach space. We frequently denote the mixed space $L^r([0,T], X)$ by $L^r_T X$.
(5) If P and Q are two operators, we denote by $[P,Q]$ the commutator operator defined by
$$[P,Q] = PQ - QP.$$
(6) $\langle .,. \rangle$ denotes the usual inner product in the Hilbert space $L^2(\mathbb{R}^2)$.
(7) For $k \in \mathbb{N}$, we denote by $C^k_B(\mathbb{R}^2)$ the space of real-valued functions $f \in C^k(\mathbb{R}^2)$ such that
$$\|f\|_{C^k_B(\mathbb{R}^2)} \equiv \sup_{\beta \in \mathbb{N}^2, |\beta| \leq k} \left\|D^\beta f\right\|_\infty < \infty.$$
We set $C^\infty_B(\mathbb{R}^2) = \cap_{k \in \mathbb{N}} C^k_B(\mathbb{R}^2)$.

2.2. **Sobolev's spaces and Chemin-Lerner's spaces.** We first recall the definition of the nonhomogeneous and homogenous Soblev's spaces.

Definition 2.1. *Let s in $\mathbb{R}$.*

(1) *The space* $H^s(\mathbb{R}^2) = H^s$ *consists of all distributions* $f \in S'(\mathbb{R}^2)$ *such that*

$$\|f\|_{H^s} \equiv \left(\int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} < \infty.$$

(2) *The space* $\dot{H}^s(\mathbb{R}^2) = \dot{H}^s$ *is the set of* $f \in S'(\mathbb{R}^2)/\mathcal{P}(\mathbb{R}^2)$ *satisfying*

$$\|f\|_{\dot{H}^s} \equiv \left(\int_{\mathbb{R}^2} |\xi|^{2s} \left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} < \infty$$

where $\mathcal{P}(\mathbb{R}^2)$ *is the space of polynomials functions defined on* $\mathbb{R}^2$.

Remark 2.1. *Let* $k \in \mathbb{N},\ s \in \mathbb{R}$ *and* $\sigma \in \mathbb{R}^+$. *By using the Plancherel formula, one can easily verify:*

$$\|f\|_{\dot{H}^s} \simeq \|\Lambda^s f\|_2 \tag{2.1}$$

$$\|f\|_{H^\sigma} \simeq \|f\|_2 + \|\Lambda^\sigma f\|_2 \tag{2.2}$$

$$\|f\|_{H^k} \simeq \sum_{\beta\in\mathbb{N}^2, |\beta|\le k} \left\|D^\beta f\right\|_2 . \tag{2.3}$$

Notation 1. *Let* $p \in \mathbb{N}\cup\{+\infty\}$ *and* $I \subset \mathbb{R}$. *In the sequel, we will use frequently the followings notations:*

$$\begin{aligned} H^\infty &\equiv \cap_{k\in\mathbb{N}} H^k \\ C^p(I, H^\infty) &\equiv \cap_{k\in\mathbb{N}} C^p(I, H^k). \end{aligned}$$

Next, we recall the notion of the Littelwood-Paley decomposition that will allow us to define the Chemin-Lerner spaces. Let $\phi \in S(\mathbb{R}^2)$ such that $\hat{\phi} \equiv 1$ on the unit ball $B(0,1)$ of $\mathbb{R}^2$ and $\hat{\phi} \equiv 0$ outside the ball $B(0,2)$. For q in $\mathbb{Z}$, we denote by S_q and Δ_q the operators defined by

$$\begin{aligned} S_q f &= \phi_q * f \\ \Delta_q f &= S_{q+1} f - S_q f \end{aligned}$$

where the star $*$ denotes the convolution on $\mathbb{R}^2$ and $\phi_q = 2^{2q}\phi(2^q.)$. For any f in $S'(\mathbb{R}^2)$, the identity, called the Littelwood-Paley decomposition of f,

$$f = \sum_{q\in\mathbb{Z}} \Delta_q f \tag{2.4}$$

holds in $S'(\mathbb{R}^2)/\mathcal{P}(\mathbb{R}^2)$. Moreover, if $f \in L^p$ with $1 \le p < \infty$ then the equality (2.4) holds in $S'(\mathbb{R}^2)$.

It is well-known that the Littlewood-Paley decomposition provides an equivalent definition to the Sobolev spaces. Namely, we have

$$\|f\|_{H^\sigma} \simeq \|S_0 f\|_2 + \left(\sum_{q\in\mathbb{N}} 2^{2\sigma q}\,\|\Delta_q f\|_2^2\right)^{1/2} \tag{2.5}$$

$$\|f\|_{\dot H^\sigma} \simeq \left(\sum_{q\in\mathbb{Z}} 2^{2\sigma q}\,\|\Delta_q f\|_2^2\right)^{1/2} \tag{2.6}$$

Now we recall the definition of Chemin-Lerner's spaces [2]

Definition 2.2. *Let $T>0,\ r\in[1,\infty]$ and $s\in\mathbb{R}$.*

(1) *The space $\tilde L^r([0,T],\dot H^s(\mathbb{R}^2))$, abbreviated by $\tilde L^r_T\dot H^s$, is the set of all tempered distribution v satisfying*

$$\|v\|_{\tilde L^r_T\dot H^s} \equiv \left(\sum_{q\in\mathbb{Z}} 2^{2sq}\,\|\Delta_q v\|^2_{L^r_T L^2}\right)^{1/2} < \infty.$$

(2) *If $s\geq 0$, we denote by $\tilde L^r([0,T],H^s(\mathbb{R}^2))$, abbreviated by $\tilde L^r_T H^s$, the space $L^r_T L^2\cap\tilde L^r_T\dot H^s$ endowed with the norm*

$$\|v\|_{\tilde L^r_T H^s} \equiv \|v\|_{L^r_T L^2} + \|v\|_{\tilde L^r_T\dot H^s}.$$

Using the estimate (2.6) and the Minkowiski inequality one can can easily obtain the following estimates

$$\|v\|_{\tilde L^r_T\dot H^s} \lesssim \|v\|_{L^r_T\dot H^s} \ \text{ if } r\leq 2 \tag{2.7}$$

$$\|v\|_{L^r_T\dot H^s} \lesssim \|v\|_{\tilde L^r_T\dot H^s} \ \text{ if } r\geq 2 \tag{2.8}$$

$$\|v\|_{L^r_T\dot H^s} \simeq \|v\|_{\tilde L^r_T\dot H^s} \ \text{ if } r=2. \tag{2.9}$$

In the sequel we will often use the following notation.

Notation 2. *Let $\sigma\geq 0$ and $T>0$. We set*

$$\mathbf{X}^\sigma_T \equiv \tilde L^\infty_T H^\sigma \cap L^2_T H^{\sigma+\frac{\alpha}{2}} \tag{2.10}$$

$$\mathbf{Z}^\sigma_T \equiv C^1([0,T],L^2)\cap C([0,T],H^\sigma)\cap L^2_T H^{\sigma+\frac{\alpha}{2}}. \tag{2.11}$$

2.3. **Preliminaries results.** In this subsection, we state and prove some elementary and useful lemmas.

The first lemma is a particular case of the well-known Bernstein inequality [9].

Lemma 2.1. *Let $\gamma\in\mathbb{R},\ \beta\in\mathbb{N}^2$ and $1\leq p\leq r\leq\infty$. The followings assertions hold true:*

(1) *There exists a constant $\nu=\nu(\gamma)>0$ such that for all $f\in S'(\mathbb{R}^2)$ and $q\in\mathbb{Z}$ we have*

$$\|\Lambda^\gamma\Delta_q f\|_p \geq \nu 2^{\gamma q}\,\|\Delta_q f\|_p. \tag{2.12}$$

(2) *There exists a constant* $C = C(\gamma, \beta, p, r) > 0$ *such that for all* $f \in S'(\mathbb{R}^2)$ *and* $q \in \mathbb{Z}$ *we have*

$$\left\|\Lambda^\gamma D^\beta \Delta_q f\right\|_r \leq C 2^{q(\gamma+|\beta|)+2q\left(\frac{1}{p}-\frac{1}{r}\right)} \left\|\Delta_q f\right\|_p . \tag{2.13}$$

The next lemma will be repeatedly used in this paper

Lemma 2.2. *Let* $\gamma \in]0,1[$, $k \in \mathbb{N}$ *and* $m \in \mathbb{N}\backslash\{0,1,2\}$. *We have the following assertions:*

(1) *For all* $f \in H^k(\mathbb{R}^2)$ *and* $g \in C_B^k(\mathbb{R}^2)$,

$$\|fg\|_{H^k} \lesssim \|g\|_{C_B^k(\mathbb{R}^2)} \|f\|_{H^k} . \tag{2.14}$$

(2) *For all* $f \in L^2(\mathbb{R}^2)$,

$$\left\|\Lambda^{\gamma-1}\mathcal{R}^\perp(f)\right\|_{\frac{2}{\gamma}} \lesssim \|f\|_2 . \tag{2.15}$$

(3) *For all* $f \in H^m(\mathbb{R}^2)$,

$$\left\|\Lambda^{\gamma-1}\mathcal{R}^\perp(f)\right\|_{C_B^{m-2}(\mathbb{R}^2)} \lesssim \|f\|_{H^m} . \tag{2.16}$$

In particular, if $f \in H^\infty(\mathbb{R}^2)$ *then* $\Lambda^\gamma \mathcal{R}^\perp(f) \in C_B^\infty(\mathbb{R}^2)$.

(4) *For all* $f \in C_B^k(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$,

$$\left\|\Lambda^{\gamma-1}\mathcal{R}^\perp(f)\right\|_{C_B^k(\mathbb{R}^2)} \lesssim \left(\|f\|_2 + \|f\|_{C_B^k(\mathbb{R}^2)}\right) . \tag{2.17}$$

Proof. (1) obvious. (2) is a direct consequence of the Sobolev embedding $\dot{H}^{1-\gamma}\left(\mathbb{R}^2\right) \hookrightarrow L^{\frac{2}{\gamma}}(\mathbb{R}^2)$ and the continuity of the Riesz transforms on $L^2(\mathbb{R}^2)$. Let us prove the assertion (3). Let $f \in H^m(\mathbb{R}^2)$ and set $g \equiv \Lambda^{\gamma-1}\mathcal{R}^\perp(f)$. According to (2.15), $g \in L^{\frac{2}{\gamma}}(\mathbb{R}^2)$. Consequentlty,

$$g = \sum_{q \in Z} \Delta_q g \text{ in } S'(\mathbb{R}^2).$$

Hence, for all $\beta \in \mathbb{N}^2$ with $|\beta| \leq m-2$, we have, thanks to (2.13),

$$\begin{aligned} \left\|D^\beta g\right\|_\infty &\lesssim \sum_{q \leq 0} 2^{q(\gamma+|\beta|)} \|\Delta_q f\|_2 + \sum_{q>0} 2^{q(\gamma-1)} \left\|\Delta_q D^\beta f\right\|_\infty \\ &\lesssim \|f\|_2 + \left\|D^\beta f\right\|_\infty \\ &\lesssim \|f\|_{H^m} . \end{aligned}$$

In the last inequality, we have used the embedding $H^m(\mathbb{R}^2) \hookrightarrow C_B^{m-2}(\mathbb{R}^2)$. The proof of the assertion (4) is similar to that of the assertion (3) and thus it is omitted. □

The next lemma is classical. For the proof see for instance [12] or [9]

Lemma 2.3. *Let σ_1 and σ_2 be two real numbers such that $\sigma_1 < 1, \sigma_2 < 1$ and $\sigma_1 + \sigma_2 > 0$. Then there exists a constant $C = C_{\sigma_1,\sigma_2} \geq 0$ such that for all $f \in \dot{H}^{\sigma_1}(\mathbb{R}^2)$ and $g \in \dot{H}^{\sigma_2}(\mathbb{R}^2)$ we have*

$$\|fg\|_{\dot{H}^{\sigma}} \leq C\,\|f\|_{\dot{H}^{\sigma_1}}\,\|g\|_{\dot{H}^{\sigma_2}} \tag{2.18}$$

where $\sigma = \sigma_1 + \sigma_2 - 1$.

The following result on differential inequalities will be useful.

Lemma 2.4. *Let $T > 0$ and $h, f : [0,T] \to \mathbb{R}^+$ two continuous functions. Assume that $h^2 \in C^1([0,T])$ and there exists $c \in \mathbb{R}$ such that for all t in $[0,T]$*

$$\left(h^2\right)'(t) + c\ h^2(t) \leq f(t)h(t).$$

Then

$$\forall t \in [0,T],\ \ h(t) \leq e^{-\frac{c}{2}t}h(0) + \frac{1}{2}\int_0^t e^{-c(t-s)}f(s)ds. \tag{2.19}$$

Proof. For $\varepsilon > 0$, we define the function $h_\varepsilon = \sqrt{\varepsilon + h^2}$. Clearly, h_ε is C^1 on $[0,T]$ and satisfies the differential inequality

$$\begin{aligned} h_\varepsilon' + \frac{c}{2}h_\varepsilon &\leq \frac{c\varepsilon}{2\sqrt{\varepsilon + h^2}} + \frac{fh}{2\sqrt{\varepsilon + h^2}} \\ &\leq \frac{c\sqrt{\varepsilon}}{2} + \frac{1}{2}f. \end{aligned}$$

Then for all t in $[0,T]$ we have

$$h_\varepsilon(t) \leq e^{-\frac{c}{2}t}h_\varepsilon(0) + \frac{c}{2}\sqrt{\varepsilon}\int_0^t e^{-c(t-s)}ds + \frac{1}{2}\int_0^t e^{-c(t-s)}f(s)ds.$$

Hence, we get the desired estimate (2.19) by sending $\varepsilon \to 0$. □

We now state and prove a version of the well-known maximal principle.

Lemma 2.5. *Let $T > 0$, $\theta \in C^1([0,T], L^2) \cap C([0,T], H^1)$ and $f, v \in C([0,T], L^2)$ such that*

$$\partial_t\theta + \Lambda^\alpha\theta + u\vec{\nabla}\theta = f$$

where $u = \Lambda^{\alpha-1}\mathcal{R}^\perp(v)$. Then, for all t in $[0,T]$, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_2^2 + \left\|\Lambda^{\alpha/2}\theta(t)\right\|_2^2 = \langle f(t), \theta(t)\rangle \tag{2.20}$$

and

$$\|\theta(t)\|_2 \leq \|\theta(0)\|_2 + \int_0^t \|f(\tau)\|_2\, d\tau. \tag{2.21}$$

Proof. Let t in $[0,T]$. We have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_2^2 &= \langle \partial_t\theta(t),\theta(t)\rangle \\ &= -\underbrace{\langle \Lambda^\alpha\theta(t),\theta(t)\rangle}_{=I(t)} - \underbrace{\langle u(t)\vec{\nabla}\theta(t),\theta(t)\rangle}_{=J(t)} + \langle f(t),\theta(t)\rangle.\end{aligned}$$

By Plancherel's formula,

$$I(t)=\left\|\Lambda^{\alpha/2}\theta(t)\right\|_2^2.$$

On the other hand, by virtue of the density, there exists two sequence $(v_n)_n$ and $(\theta_n)_n$ in $C_c^\infty(\mathbb{R}^2)$ such that

$$v_n \to v(t) \text{ in } L^2 \text{ and } \theta_n \to \theta(t) \text{ in } H^1.$$

Consequently, the estimate (2.15) and the Sobolev embedding $H^1 \hookrightarrow L^{\frac{2}{1-\alpha}}$ imply that

$$u_n \equiv \Lambda^{\alpha-1}\mathcal{R}^\perp(v_n) \to u \text{ in } L^{\frac{2}{\alpha}} \text{ and } \theta_n \to \theta(t) \text{ in } L^{\frac{2}{1-\alpha}}.$$

Hence, Holder's inequality yields

$$J_n \equiv \langle u_n\vec{\nabla}\theta_n,\theta_n\rangle \to J(t).$$

Now, a simple integration by parts using the fact that u_n is divergence-free gives

$$J_n = -J_n$$

Thus $J_n=0$ and consequently $J(t)=0$. This finishes the proof of (2.20). Finally, the estimate (2.21) is an immediate consequence of (2.20) and Lemma 2.5. □

The following lemma is simple, hence its proof is omitted.

Lemma 2.6. *Let $T>0$, $\sigma\in\mathbb{R}$ and $r\in[1,\infty]$. Assume that a sequence $(\theta_n)_n$ converges to a function θ in the space $L^\infty([0,T],L^2(\mathbb{R}^2))$ and that*

$$\mathcal{M}\equiv\sup_n\|\theta_n\|_{\tilde{L}^r_T\dot{H}^\sigma}<\infty.$$

Then $\theta\in\tilde{L}^r_T\dot{H}^\sigma$ and satisfies

$$\|\theta\|_{\tilde{L}^r_T\dot{H}^\sigma}\le\mathcal{M}.$$

Lemma 2.7. *Let $T>0$ and $\sigma\in\mathbb{R}$. If a function θ belongs to the spaces $C([0,T],L^2(\mathbb{R}^2))$ and $\tilde{L}^\infty\left([0,T],\dot{H}^\sigma\right)$ then it belongs to the space $C\left([0,T],\dot{H}^\sigma\right)$.*

Proof. For $N \in \mathbb{N}$ define $\theta_N = \sum_{|j|\leq N} \Delta_j \theta$. The sequence $(\theta_N)_N$ is in the space $C\left([0,T], \dot{H}^\sigma\right)$; in fact, for every $t, t' \in [0,T]$ we have

$$\begin{aligned} \left\|\theta_N(t) - \theta_N(t')\right\|^2_{\dot{H}^\sigma} &= \sum_{|j|\leq N} 2^{2\sigma j} \left\|\Delta_j \left[\theta(t) - \theta(t')\right]\right\|_2^2 \\ &\leq \left\|\theta(t) - \theta(t')\right\|_2 \sum_{|j|\leq N} 2^{2\sigma j}. \end{aligned}$$

On the other hand, $(\theta_N)_N$ converges to θ in the space $L^\infty\left([0,T], \dot{H}^\sigma\right)$; indeed, for every t in the interval $[0,T]$ we have

$$\|\theta(t) - \theta_N(t)\|^2_{\dot{H}^\sigma} \leq 4 \sum_{|j|>N} 2^{2\sigma j} \left\|\Delta_j \theta\right\|^2_{L^\infty_T L^2}$$

and $\sum_{|j|>N} 2^{2\sigma j} \|\Delta_j\theta\|^2_{L^\infty_T L^2} \to 0$ as $N \to \infty$ since $\theta \in \tilde{L}^\infty\left([0,T], \dot{H}^\sigma\right)$. Finally, since $C\left([0,T], \dot{H}^\sigma\right)$ is closed sub-space of $L^\infty\left([0,T], \dot{H}^\sigma\right)$, we conclude that θ is in $C\left([0,T], \dot{H}^\sigma\right)$. □

Now we state and prove the main result of this sub-section.

Proposition 2.1. *Let $\alpha \in]0,1[$, $T > 0$, $\theta_* \in H^\infty(\mathbb{R}^2)$ and $v \in C([0,T], H^\infty(\mathbb{R}^2))$. Set $u = \Lambda^{\alpha-1}\mathcal{R}^\perp(v)$. Then there is a unique solution $\theta \in C^1([0,T], H^\infty(\mathbb{R}^2))$ to the linear initial value problem*

$$\text{(IVPL)} \qquad \begin{cases} \partial_t\theta + \Lambda^\alpha\theta + u\vec{\nabla}\theta = 0 \\ \theta|_{t=0} = \theta_*. \end{cases}$$

Moreover,

$$\sup_{0\leq t\leq T} \|\theta(t)\|_2 \leq \|\theta_*\|_2 \tag{2.22}$$

and, for every $r \geq 2$ and $s \geq 0$ there exists a constant $\mathcal{C}_{r,s} > 0$ depending only on r and s such that

$$\|\theta\|_{\tilde{L}^r_T \dot{H}^{s+\frac{\alpha}{r}}} \leq \mathcal{K}_{r,s}(\theta_*, T) + \mathcal{C}_{r,s}\left(\|v\|_{L^2_T \dot{H}^{1+\frac{\alpha}{2}}} \|\theta\|_{\tilde{L}^r_T \dot{H}^{s+\frac{\alpha}{r}}} + \|\theta\|_{L^2_T \dot{H}^{1+\frac{\alpha}{2}}} \|v\|_{\tilde{L}^r_T \dot{H}^{s+\frac{\alpha}{r}}}\right) \tag{2.23}$$

where

$$\mathcal{K}_{r,s}(\theta_*, T) = \left\|\left(\frac{1 - e^{-\nu r 2^{\alpha q}T}}{\nu r}\right)^{1/r} 2^{sq}\left\|\Delta_q\theta_*\right\|_2\right\|_{l^2(\mathbb{Z})} \quad \text{if } r < \infty, \tag{2.24}$$

and

$$\mathcal{K}_{\infty,s}(\theta_*, T) = \left\|2^{sq}\left\|\Delta_q\theta_*\right\|_2\right\|_{l^2(\mathbb{Z})}. \tag{2.25}$$

ν is the real given by Lemma 2.2.

The proof of this proposition is based on the following commutator estimate which can be easily proved by following the arguments used for instance in [1], [3], [7], [11] or [13] .

Lemma 2.8 (Commutator estimate)**.** *Let* $\alpha \in]0,1[$, $T>0$ *and* f,g *two functions in the space* $C([0,T],H^{\infty}(\mathbb{R}^2))$. *Set* $u=\Lambda^{\alpha-1}\mathcal{R}^{\perp}(f)$. *Then for all* $r\geq 2$ *and* $s\geq 0$ *there exists a constant* $C_{r,s}$ *depending only on* r *and* s *and a sequence* $(\varepsilon_q)_q \in l^2(\mathbb{Z})$ *with* $\sum_q \varepsilon_q^2 \leq 1$ *such that for all* $q\in\mathbb{Z}$ *we have*

(2.26)
$$\left\|[\Delta_q,u]\,\vec{\nabla}g\right\|_{L^{\rho}_T L^2} \leq C_{r,s}2^{-q(s+\frac{\alpha}{r}-\frac{\alpha}{2})}\varepsilon_q\left(\|f\|_{L^2_T\dot{H}^{1+\frac{\alpha}{2}}}\|g\|_{\tilde{L}^r_T\dot{H}^{s+\frac{\alpha}{r}}}+\|g\|_{L^2_T\dot{H}^{1+\frac{\alpha}{2}}}\|f\|_{\tilde{L}^r_T\dot{H}^{s+\frac{\alpha}{r}}}\right)$$

where $\frac{1}{\rho}=\frac{1}{2}+\frac{1}{r}$.

Proof of Proposition 3. We first notice that thanks to Lemma 2.2, the function u belongs to the space $\cap_{k\in\mathbb{N}}C([0,T],C^k_B(\mathbb{R}^2))$. Hence, for all $m\in\mathbb{N}$,

$$\mathcal{M}_m(u)\equiv\sup_{|\beta|\leq m}\left\|D^{\beta}u\right\|_{L^{\infty}([0,T]\times\mathbb{R}^2)}<\infty. \tag{2.27}$$

Let us prove the existence of the solution θ. To do so, we will make use of the classical Fredireck's method. For $n\in\mathbb{N}$, we consider the linear ODE

$$\left\{\begin{array}{l}\partial_t\theta=F_n(t,\theta)\\ \theta|_{t=0}=J_n\theta_*\end{array}\right. \tag{S$_n$}$$

where the operator J_n is defined by

$$\widehat{J_nf}(\xi)=1_{B(0,n)}(\xi)\ \hat{f}(\xi)$$

and $F_n:[0,T]\times L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ is the function defined by

$$F_n\left(t,g\right)\equiv-\Lambda^{\alpha}J_ng-J_n(u\vec{\nabla}J_n\theta).$$

Clearly, F_n belongs to $C([0,T],\mathcal{L}\left(L^2(\mathbb{R}^2)\right))$. Hence the Cauchy-Lipschitz Theorem ensures that (S$_n$) has a unique global solution θ_n belonging to the space $C^1([0,T],L^2(\mathbb{R}^2))$. Now, since $J_n^2=J_n$, then $J_n\theta_n$ is also a solution to (S$_n$). Therefore, $\theta_n=J_n\theta_n$ which implies in particular that $\theta_n\in C^1([0,T],H^{\infty}(\mathbb{R}^2))$. Now we will estimate the norm of θ_n in the spaces $L^{\infty}_TH^m$ where $m\in\mathbb{N}$. Let $m\in\mathbb{N}$ and $\beta\in\mathbb{N}^2$ such that $|\beta|\leq m$. Clearly, the function $D^{\beta}\theta_n$ satisfies the equation

$$\partial_tD^{\beta}\theta_n+\Lambda^{\alpha}D^{\beta}\theta_n+u\vec{\nabla}D^{\beta}\theta_n=J_n\left(\left[u,D^{\beta}\right]\vec{\nabla}\theta_n\right)+\tilde{J}_n\left(u\vec{\nabla}D^{\beta}\theta_n\right)$$

where $\tilde{J}_n$ is defined by

$$\widehat{\tilde{J}_nf}(\xi)=\left(1-1_{B(0,n)}(\xi)\right)\ \hat{f}(\xi). \tag{2.28}$$

Hence, Lemma 2.5 implies

$$\begin{aligned}\frac{d}{2dt}\left\|D^\beta\theta_n(t)\right\|_2^2 &\leq \langle J_n\left(\left[u,D^\beta\right]\vec{\nabla}\theta_n\right),D^\beta\theta_n\rangle+\langle\tilde{J}_n\left(u\vec{\nabla}D^\beta\theta_n\right),D^\beta\theta_n\rangle\\ &= \langle\left[u,D^\beta\right]\vec{\nabla}\theta_n,D^\beta\theta_n\rangle\\ &\leq \left\|\left[u,D^\beta\right]\vec{\nabla}\theta_n\right\|_2\left\|D^\beta\theta_n(t)\right\|_2^2\\ &\leq C_\beta\,\mathcal{M}_m(u)\,\|\theta_n(t)\|_{H^m}^2\end{aligned}$$

where in the second equation we have used the fact $J_nD^\beta\theta_n=D^\beta\theta_n$ and $\tilde{J}_nD^\beta\theta_n=0$. We then deduce that there exists a constant $C_m>0$ depending only on m, such that

$$\frac{d}{dt}\|\theta_n(t)\|_{H^m}^2\leq C_m\mathcal{M}_m(u)\,\|\theta_n(t)\|_{H^m}^2$$

Invoking Gronwall's inequality, we then infer that the sequence $(\theta_n)_n$ is bounded in the space $C([0,T],H^m(\mathbb{R}^2))$. That is,

$$\forall m\in\mathbb{N},\ \Gamma_m\equiv\sup_n\|\theta_n\|_{L_T^\infty H^m}<\infty. \tag{2.29}$$

Next we will show that $(\theta_n)_n$ is of Cauchy in the space $C([0,T],L^2(\mathbb{R}^2))$. Let $p,q\in\mathbb{N}$ such that $p\leq q$. The function $\omega=\theta_q-\theta_p$ satisfies the equation

$$\partial_t\omega+\Lambda^\alpha\omega+u\vec{\nabla}\omega=(J_q-J_p)\left(u\vec{\nabla}\theta_p\right)+\tilde{J}_q\left(u\vec{\nabla}\omega\right)$$

where $\tilde{J}_q$ is defined by (2.28). Using Lemma 2.5 and the fact that $\tilde{J}_q\omega=0$, we easily get the following estimates

$$\begin{aligned}\frac{d\,\|\omega(t)\|_2^2}{2dt} &\leq \left\|(J_q-J_p)\left(u\vec{\nabla}\theta_p\right)\right\|_2\|\omega(t)\|_2\\ &\lesssim \frac{1}{p}\left\|\left(u\vec{\nabla}\theta_p\right)\right\|_{H^1}\|\omega(t)\|_2\\ &\lesssim \frac{1}{p}\mathcal{M}_2(u)\,\|\theta_p\|_{H^2}\,\|\omega(t)\|_2\\ &\lesssim \frac{1}{p}\|\omega(t)\|_2\,.\end{aligned}$$

In the last inequality, we used (2.27)-(2.29). Hence, Lemma 2.4 implies that there exists a constant C independent on p and q such that

$$\sup_{0\leq t\leq T}\|\theta_q(t)-\theta_p(t)\|_2\leq\|(J_q-J_p)\,\theta_*\|_2+\frac{C}{p},$$

which leads the required result. Let θ be the limit of the sequence $(\theta_n)_n$ in the space $C([0,T],L^2(\mathbb{R}^2))$. Now thanks to the interpolation inequality

$$\|f\|_{H^m}\leq C_m\sqrt{\|f\|_2\,\|f\|_{H^{2m}}}$$

and the uniform boundness (2.29), we infer that the sequence $(\theta_n)_n$ converges to θ in the space $C([0,T],H^m(\mathbb{R}^2))$ for all m in $\mathbb{N}$. Moreover, since

$$\partial_t\theta_n = -\Lambda^\alpha\theta_n - J_n\left(u\vec{\nabla}\theta_n\right)$$

then by using the first assertion of Lemma 2.1 and the continuity of the operators Λ^α and $\vec{\nabla}$ from H^m into H^{m-1}, we easily deduce that, for all for all m in $\mathbb{N}$, the sequence $(\partial_t\theta_n)_n$ converges to $-\Lambda^\alpha\theta - u\vec{\nabla}\theta$ in the space $C([0,T],H^m(\mathbb{R}^2))$. We therefore conclude that θ belongs to $C^1([0,T],H^\infty(\mathbb{R}^2))$ and it is a solution to (IVPL).The uniqueness can be easily proved, indeed if θ' is another solution to (IVPL) then the difference function $\delta = \theta - \theta'$ satisfies

$$\begin{cases} \partial_t\delta + \Lambda^\alpha\delta + u\vec{\nabla}\delta = 0 \\ \delta|_{t=0} = 0, \end{cases}$$

which en virtue of Lemma (2.5) implies $\delta = 0$ and consequently, $\theta = \theta'$.

Once again the estimate (2.22) is a consequence of Lemma 2.5. Finally, let us prove the estimate (2.23) in the case $r \in [2,\infty[$, the proof in the case $r=\infty$ is similar and even more simpler.

Apply the operator Δ_q $(q \in \mathbb{Z})$ to the first equation of (IVPL), we get

$$\partial_t\Delta_q\theta + \Lambda^\alpha\Delta_q\theta + u\vec{\nabla}\Delta_q\theta = [u,\Delta_q]\,\vec{\nabla}\theta.$$

Therefore, Lemma 2.5 implies

$$\frac{d}{2dt}\left\|\Delta_q\theta(t)\right\|_2^2 + \left\|\Lambda^{\alpha/2}\Delta_q\theta(t)\right\|_2^2 \le F_q(t)\left\|\Delta_q\theta(t)\right\|_2$$

where

$$F_q(t) = \left\|[u,\Delta_q]\,\vec{\nabla}\theta\right\|_2.$$

Thanks to the first assertion of Lemma 2.1, we deduce that there exists a pure constant $\nu > 0$ such that

$$\frac{d}{2dt}\left\|\Delta_q\theta(t)\right\|_2^2 + \nu 2^{\alpha q}\left\|\Delta_q\theta(t)\right\|_2^2 \le F_q(t)\left\|\Delta_q\theta(t)\right\|_2.$$

Invoking Lemma 2.4, we obtain

$$\left\|\Delta_q\theta(t)\right\|_2 \le e^{-\nu 2^{\alpha q}t}\left\|\Delta_q\theta_*\right\|_2 + \int_0^t e^{-\nu 2^{\alpha q}(t-s)}F_q(s)ds.$$

Let $r \in [2,\infty[$ and set $\rho = (\frac{1}{2}+\frac{1}{r})^{-1}$. Using the Young inequality, we deduce from the above inequality that

$$\begin{aligned} \left\|\Delta_q\theta\right\|_{L^r_TL^2} &\le \left(\frac{1-e^{-\nu 2^{\alpha q}T}}{\nu r}\right)^{1/r} 2^{-\frac{\alpha}{r}q}\left\|\Delta_q\theta_*\right\|_2 + \left\|e^{-\nu 2^{\alpha q}t}\right\|_{L^2(\mathbb{R}^+)}\left\|F_q\right\|_{L^\rho([0,T])} \\ &\le \left(\frac{1-e^{-\nu 2^{\alpha q}T}}{\nu r}\right)^{1/r} 2^{-\frac{\alpha}{r}q}\left\|\Delta_q\theta_*\right\|_2 + C_\nu 2^{-\frac{\alpha}{2}q}\left\|F_q\right\|_{L^\rho([0,T])}. \end{aligned}$$

Multiplying the both sides of the last inequality by $2^{(s+\frac{\alpha}{r})q}$, using the commutator estimate (2.26) and then taking the $l^2(\mathbb{Z})$ norm, we obtain the desired estimate (2.23). $\square$

Lemma 2.9. *Let $(x_n)_{n\in\mathbb{N}}$ be a non negative real sequence. Assume there exists two constants A and $B \geq 0$ such that*

$$\begin{cases} x_0 \leq 2A \\ \forall n,\ x_{n+1} \leq A + Bx_n x_{n+1}. \end{cases}$$

If $4AB \leq 1$, then

$$\forall n,\ x_n \leq 2A. \tag{2.30}$$

Proof. This lemma can be easily proved by induction. □

Lemma 2.10. *Let $(x_n)_{n\in\mathbb{N}}$ be a non negative real sequence. Assume there exists two constants A and $\delta \geq 0$ such that*

$$\forall n,\ x_{n+1} \leq A + \delta x_n.$$

If $\delta < 1$, then the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded.

Proof. Obvious. In fact, for all n we have

$$\begin{aligned} x_n &\leq A\sum_{k=0}^{n} \delta^k + \delta^n x_0 \\ &\leq \frac{A}{1-\delta} + x_0. \end{aligned}$$

□

Lemma 2.11. *let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a normed vectorial space $(E, \|.\|)$ Assume, there exists a real sequence $\left(\gamma_{n,p}\right)_{n,p}$ and a real number δ such that for all (n,p) in $\mathbb{N}^2$,*

$$\|x_{n+1+p} - x_{n+1}\| \leq \gamma_{n,p} + \delta \|x_{n+p} - x_n\|. \tag{2.31}$$

If

$$\delta \in [0,1[\text{ and } \left(\sup_{p\in\mathbb{N}} \gamma_{n,p}\right)_n \to 0 \text{ as } n \to \infty$$

then, the sequence $(x_n)_n$ is of Cauchy in $(E, \|.\|)$.

Proof. Define $u_n = \sup_{p\in\mathbb{N}} \|x_{n+p} - x_n\|$ and $\gamma_n = \sup_{p\in\mathbb{N}} \gamma_{n,p}$. According to (2.31), we have

$$\forall n,\ u_{n+1} \leq \gamma_n + \delta u_n.$$

This implies

$$\forall n,\ u_n \leq \sum_{k=0}^{n} \delta^k \gamma_{n-k} + \delta^n u_0.$$

Hence, for all $n > n_0$ in $\mathbb{N}$, we have

$$u_n \leq \frac{1}{1-\delta}\left(\sup_{0\leq k\leq n_0} \gamma_{n-k}\right) + \left(\sup_{k\geq 0} \gamma_k\right)\frac{\delta^{n_0+1}}{1-\delta} + \delta^n u_0.$$

Fixing n_0 and taking the $\overline{\lim}_n$, we obtain

$$\overline{\lim_n}\ u_n \le \left(\sup_{k\ge 0}\gamma_k\right)\frac{\delta^{n_0+1}}{1-\delta}.$$

Letting $n_0 \to +\infty$, we get

$$\overline{\lim_n}\ u_n = 0.$$

That is $(x_n)_n$ is a Cauchy sequence in $(E, \|.\|)$. □

3. Proof of Proposition 1.1

This section is devoted to the proof of Proposition 1.1. This proof is motivated by the work [3].

Proof of Proposition 1.1. Set $\theta_0 \equiv 0$. Proposition 2.1 enables us to construct by induction the sequence of functions $\theta_n \in C^1(\mathbb{R}^+, H^\infty)$ solutions to the systems

$$\begin{cases} \partial_t\theta_{n+1} + \Lambda^\alpha\theta_{n+1} + u_n\vec{\nabla}\theta_{n+1} = 0 \\ \theta_{n+1}|_{t=0} = S_{n+1}\theta_* \\ u_n = \Lambda^{\alpha-1}\mathcal{R}^\perp(\theta_n) \end{cases}$$

and satisfying, for all $r \ge 2,\ s \ge 0$ and $T > 0$, the following estimates:

$$\sup_{t\ge 0}\|\theta_n(t)\|_2 \le \|\theta_*\|_2 \tag{3.1}$$

(3.2)

$$\|\theta_{n+1}\|_{\tilde{L}^r_T\dot{H}^{s+\frac{\alpha}{r}}} \le \mathcal{K}_{r,s}(\theta_*,T) + \mathcal{C}_{r,s}\left(\|\theta_n\|_{L^2_T\dot{H}^{1+\frac{\alpha}{2}}}\|\theta_{n+1}\|_{\tilde{L}^r_T\dot{H}^{s+\frac{\alpha}{r}}} + \|\theta_{n+1}\|_{L^2_T\dot{H}^{1+\frac{\alpha}{2}}}\|\theta_n\|_{\tilde{L}^r_T\dot{H}^{s+\frac{\alpha}{r}}}\right)$$

where $\mathcal{K}_{r,s}(\theta_*,T)$ is defined by (2.24)-(2.25) and $\mathcal{C}_{r,s}$ is a constant depending only on r and s.

Let $\varepsilon_\sigma > 0$ to be determined later and assume that for some $T > 0$ we have

$$\mathcal{K}(\theta_*,T) = \mathcal{K}_{2,1}(\theta_*,T) \le \varepsilon_\sigma,$$

(T exists since $\mathcal{K}(\theta_*,T) \to 0$ as $T \to 0^+$). Then, applying the estimate (3.2) with $s = 1$ and $r = 2$ yields

$$\|\theta_{n+1}\|_{L^2_T\dot{H}^{1+\frac{\alpha}{2}}} \le \varepsilon_\sigma + 2\mathcal{C}_{2,1}\|\theta_n\|_{L^2_T\dot{H}^{1+\frac{\alpha}{2}}}\|\theta_{n+1}\|_{L^2_T\dot{H}^{1+\frac{\alpha}{2}}}.$$

Hence, according to Lemma 2.9, we have

$$\forall n,\ \|\theta_n\|_{L^2_T\dot{H}^{1+\frac{\alpha}{2}}} \le 2\varepsilon_\sigma \tag{3.3}$$

provided that

$$8\varepsilon_\sigma\mathcal{C}_{2,1} \le 1. \tag{3.4}$$

Using now the estimate (3.2) with $s = \sigma$ and $r \in \{2, +\infty\}$, we get

$$\|\theta_{n+1}\|_{\tilde{L}^r_T\dot{H}^{\sigma+\frac{\alpha}{r}}} \le \mathcal{K}_{r,s}(\theta_*,T) + 2\varepsilon_\sigma\mathcal{C}_{r,\sigma}\left(\|\theta_n\|_{\tilde{L}^r_T\dot{H}^{\sigma+\frac{\alpha}{r}}} + \|\theta_{n+1}\|_{\tilde{L}^r_T\dot{H}^{\sigma+\frac{\alpha}{r}}}\right).$$

Therefore, Lemma 2.10 ensures that the sequence $(\theta_n)_n$ is bounded in the spaces $\tilde{L}^2_T\dot{H}^{\sigma+\frac{\alpha}{2}}$ and $\tilde{L}^\infty_T\dot{H}^\sigma$ provided that

$$4\varepsilon_\sigma \max\{\mathcal{C}_{2,\sigma},\mathcal{C}_{\infty,\sigma}\} \leq 1. \tag{3.5}$$

We claim now that if ε_σ is small enough then the sequence $(\theta_n)_n$ is of Cauchy in the space $C([0,T],L^2(\mathbb{R}^2))$. Let $n\in\mathbb{N}$ and $p\in\mathbb{N}^*$. Define $\omega_{n+1}=\theta_{n+1+p}-\theta_{n+1}$, $\omega_n=\theta_{n+p}-\theta_n$ and $v_n=\Lambda^{\alpha-1}\mathcal{R}^\perp(\omega_n)$. Clearly, we have the equation

$$\partial_t\omega_{n+1}+\Lambda^\alpha\omega_{n+1}+u_{n+p}\vec{\nabla}\omega_{n+1}+v_n\vec{\nabla}\theta_{n+1}=0.$$

Hence, Lemma 2.5 implies that for all t in $[0,T]$,

$$\frac{d}{2dt}\left\|\omega_{n+1}(t)\right\|_2^2+\left\|\Lambda^{\alpha/2}\omega_{n+1}(t)\right\|_2^2\leq\kappa_n(t)\equiv\left|\langle v_n\vec{\nabla}\theta_{n+1},\omega_{n+1}\rangle\right|. \tag{3.6}$$

Now a simple calculation gives the following estimates

$$\begin{aligned}
\kappa_n(t) &\leq \left\|v_n\vec{\nabla}\theta_{n+1}\right\|_{\dot{H}^{-\alpha/2}}\left\|\omega_{n+1}\right\|_{\dot{H}^{\alpha/2}} && (3.7)\\
&\lesssim \left\|v_n(t)\right\|_{\dot{H}^{1-\alpha}}\left\|\vec{\nabla}\theta_{n+1}(t)\right\|_{\dot{H}^{\alpha/2}}\left\|\Lambda^{\alpha/2}\omega_{n+1}(t)\right\|_2 && (3.8)\\
&\lesssim \left\|\omega_n(t)\right\|_2\left\|\theta_{n+1}(t)\right\|_{\dot{H}^{1+\alpha/2}}\left\|\Lambda^{\alpha/2}\omega_{n+1}(t)\right\|_2 \\
&\leq C\left\|\theta_{n+1}(t)\right\|_{\dot{H}^{1+\alpha/2}}^2\left\|\omega_n(t)\right\|_2^2+\left\|\Lambda^{\alpha/2}\omega_{n+1}(t)\right\|_2^2 && (3.9)
\end{aligned}$$

where to obtain (3.8) from (3.7) we have used Lemma (2.3) with $\sigma_1=1-\alpha$ and $\sigma_2=\alpha/2$.

Substituting (3.9) in (3.6), integrating with respect to time and taking the supremum over all $t\in[0,T]$ yield

$$\sup_{0\leq t\leq T}\left\|\omega_{n+1}(t)\right\|_2^2\leq\left\|\omega_{n+1}(0)\right\|_2^2+C\sup_{0\leq t\leq T}\left\|\omega_{n+1}(t)\right\|_2^2\int_0^T\left\|\theta_{n+1}(t)\right\|_{\dot{H}^{1+\alpha/2}}^2dt.$$

Recalling the estimate (3.3) and using the the fact

$$\forall a,b\in\mathbb{R}^+,\ \sqrt{a^2+b^2}\leq a+b,$$

we infer from the above inequality that

$$\left\|\theta_{n+1+p}-\theta_{n+1}\right\|_*\leq\delta_{n,p}+C\varepsilon_\sigma\left\|\theta_{n+p}-\theta_n\right\|_* \tag{3.10}$$

where $\|.\|_*$ denotes the norm of the space $L^\infty([0,T],L^2(\mathbb{R}^2))$ and

$$\delta_{n,p}\equiv\left\|S_{n+p+1}\theta_*-S_{n+1}\theta_*\right\|_2.$$

Now since $\theta_*\in L^2(\mathbb{R}^2)$ then $\left(\sup_p\delta_{n,p}\right)_n\to0$ as $n\to+\infty$. Hence, according to Lemma 2.11, the sequence $(\theta_n)_n$ is of Cauchy in the space $C([0,T],L^2(\mathbb{R}^2))$ provided

$$C\varepsilon_\sigma<1. \tag{3.11}$$

In conclusion, there exists a constant $\varepsilon_\sigma>0$ depending only on σ such that if $\mathcal{K}(\theta_*,T)\leq\varepsilon_\sigma$ for some $T>0$ then the sequence $(\theta_n)_n$ is bounded in the space $\mathbf{X}^\sigma_T\equiv\tilde{L}^\infty_TH^\sigma\cap L^2_TH^{\sigma+\frac{\alpha}{2}}$ and

converges in the space $C([0,T],L^2(\mathbb{R}^2))$ to a function θ. Hence, according to Lemma 2.6 and Lemma 2.7, the function θ belongs to the space $\mathbf{X}_T^\sigma \cap C([0,T],H^\sigma(\mathbb{R}^2))$. On the other hand, using the embedding $\tilde{L}_T^\infty H^\sigma \subset L_T^\infty H^\sigma$ and the interpolation inequality

$$\forall 0 \leq s \leq \sigma,\ \|f\|_{\dot{H}^s} \leq \|f\|_2^{1-\frac{s}{\sigma}} \|f\|_{\dot{H}^\sigma}^{\frac{s}{\sigma}},$$

we deduce that $(\theta_n)_n$ converges to θ in $C([0,T],\dot{H}^{\sigma'}(\mathbb{R}^2))$ for all $\sigma' < \sigma$ which implies in particular that

$$\begin{array}{rcl} \Lambda^\alpha \theta_n & \to & \Lambda^\alpha \theta \text{ in } C([0,T],L^2(\mathbb{R}^2)) \\ \theta_n & \to & \theta \text{ in } C([0,T],\dot{H}^{\sigma_*}(\mathbb{R}^2)) \\ u_n & \to & u \equiv \Lambda^{\alpha-1}\mathcal{R}^\perp(\theta) \text{ in } C([0,T],\dot{H}^{\sigma_*+1-\alpha}(\mathbb{R}^2)) \end{array}$$

where $\sigma_* = \frac{1+\alpha}{2}$. Thus, using the fact that $u_n\vec{\nabla}\theta_{n+1} = \nabla.(\theta_{n+1}u_n)$ and Lemma 2.3 with $\sigma_1 = \sigma_* + 1 - \alpha$ and $\sigma_2 = \sigma_*$, we deduce that $u_n\vec{\nabla}\theta_{n+1}$ converges to $u\vec{\nabla}\theta$ in $C([0,T],L^2(\mathbb{R}^2))$. We then conclude that the function θ belongs to the space $\mathbf{Z}_T^\sigma$ and satisfies

$$\begin{cases} \partial_t\theta + \Lambda^\alpha\theta + u\vec{\nabla}\theta = 0 \\ u = \Lambda^{\alpha-1}\mathcal{R}^\perp(\theta) \\ \theta|_{t=0} = \theta_*. \end{cases}$$

Finally, it remains to prove the uniqueness. Assume that θ_a and $\theta_b \in \mathbf{Z}_T^\sigma$ are two solutions to the equation (MQG) with the same data θ_*. Set $u_a = \Lambda^{\alpha-1}\mathcal{R}^\perp(\theta_a),\ u_b = \Lambda^{\alpha-1}\mathcal{R}^\perp(\theta_b)$, $\omega = \theta_a - \theta_b$ and $u = u_a - u_b$. We have the equation

$$\partial_t\omega + \Lambda^\alpha\omega + u_a\vec{\nabla}\omega + u\vec{\nabla}\theta_b = 0$$

which implies, by Lemma 2.5,

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_2^2 + \left\|\Lambda^{\alpha/2}\omega(t)\right\|_2^2 \leq \kappa(t) \equiv \left|\int_{\mathbb{R}^2} u\vec{\nabla}\theta_b\omega dx\right|. \tag{3.12}$$

Following the same argument leading to (3.9), we obtain

$$\kappa(t) \leq C\|\theta_b(t)\|_{\dot{H}^{1+\alpha/2}}^2 \|\omega(t)\|_2^2 + \left\|\Lambda^{\alpha/2}\omega(t)\right\|_2^2.$$

Inserting this estimate in the inequality (3.12), we get

$$\frac{d}{dt}\|\omega(t)\|_2^2 \lesssim \|\theta_b(t)\|_{\dot{H}^{1+\alpha/2}}^2 \|\omega(t)\|_2^2.$$

Thereby, the Gronwall inequality implies $\omega = 0$ that is $\theta_a = \theta_b$. This completes the proof of the proposition. $\square$

4. Proof of Proposition 1.2

The proof of Proposition 1.2 is based on the following blowup criterion.

Lemma 4.1. *Let $\theta_* \in H^\sigma(\mathbb{R}^2)$ with $\sigma \geq 1$ and let*

$$\theta \in \bigcap_{T<T^*} \mathbf{Z}_T^\sigma$$

be the maximal solution to the equation (MQG) with initial data θ_*. Assume $T^* < \infty$. Then there is a constant $c_\sigma > 0$ depending only on σ such that

$$\forall 0 \leq t < T^*,\ \|\theta(t)\|_{\dot{H}^{1+\frac{\alpha}{2}}} \geq \frac{c_\sigma}{\sqrt{T^*-t}}. \tag{4.1}$$

In particular,

$$\int_0^{T^*} \|\theta(t)\|^2_{\dot{H}^{1+\frac{\alpha}{2}}}\, dt = +\infty.$$

Proof. Let t in $[0, T^*[$. Firstly, according to the last assertion of Proposition 1.1, we must have

$$\mathcal{K}(\theta(t), T^*-t) \geq \varepsilon_\sigma. \tag{4.2}$$

Secondly, for every $f \in H^{1+\frac{\alpha}{2}}$ and $T>0$, we have the following estimates

$$\begin{aligned}\mathcal{K}(f,T) &\leq \sqrt{T}\sup_{q\in\mathbb{Z}}\left(\frac{1-e^{-2\nu 2^{\alpha q}T}}{2\nu 2^{\alpha q}T}\right)^{1/2}\left\|\left(2^{(1+\frac{\alpha}{2})q}\,\|\Delta_q f\|_2\right)_q\right\|_{l^2(\mathbb{Z})}\\ &\leq C_*\sqrt{T}\,\|f\|_{\dot{H}^{1+\frac{\alpha}{2}}}\end{aligned} \tag{4.3}$$

where $C_* = \sup_{x>0}\sqrt{\frac{1-e^{-x}}{x}}$. Hence, combining the estimates (2.14)-(2.15) yields (4.1) with $c_\sigma = \frac{\varepsilon_\sigma}{C_*}$. ∎

Now we are in position to prove our proposition:

Proof of Proposition 1.2. In view of Proposition 1.1, the modified quasi-geostrophic equation corresponding to the initial data $\tilde{\theta}_* \equiv \theta(t_0)$ has a unique maximal solution

$$\tilde{\theta} \in \cap_{T<T^*}\mathbf{Z}_T^\sigma.$$

On the other hand, one can easily verify that the function $\theta_{t_0} \equiv \theta(.+t_0)$ is also a solution to the same (MQG) equation. Hence uniqueness in the space $\mathbf{Z}_T^1$ implies that $\theta_{t_0} = \tilde{\theta}$ on the interval $[0, \tau_*[$ where $\tau_* = \inf\{T^*; T_*-t_0\}$. Therefore, since θ_{t_0} belongs to the space $L^2([0, T_*-t_0[, H^{1+\frac{\alpha}{2}})$ then the preceding lemma ensures that $T^* > T_* - t_0$. Thus we deduce that $\theta_{t_0} \in \mathbf{Z}^\sigma_{T_*-t_0}$ which implies the required result. ∎

5. Proof of the main Theorem

Firstly, Proposition 1.1 ensures the existence of a unique maximal solution $\theta \in \cap_{T<T^*}\mathbf{Z}_T^\sigma$ to the equation (MQG). Now let $a<T$ be a two fixed real-number in the interval $]0,T^*[$. successive application of Proposition 2, allows us to construct an increasing sequence $(t_n)_{n\in\mathbb{N}} \in]0,a[$ such that for all n, $\theta(t_n)\in H^{\sigma_n}$ and

$$\theta \in C([t_n,T],H^{\sigma_n})\cap L^2([t_n,T];H^{\sigma_{n+1}})$$

where $\sigma_k \equiv \sigma + k\frac{\alpha}{2}$. Consequently, the solution θ belongs to the space $C([a,T],H^\infty)$. Now, since a and T are arbitrary chosen in $]0,T^*[$ then θ is in the space $C\left(]0,T^*[,H^\infty\right)$. On the other hand, the equation

$$\partial_t\theta = -\Lambda^\alpha\theta - u\vec{\nabla}\theta,$$

combined with the continuity Λ^α and $\vec{\nabla}$ on the space H^∞ and Lemma 2.1, enable us, via a standard Boot-strap argument, to convert the space regularity of θ to time regularity. We then deduce that

$$\theta \in C^\infty\left(]0,T^*[,H^\infty\right).$$

Now, we will establish that the solution θ is global in time, that is $T^*=\infty$. We will argue by opposition: we suppose that $T^*<\infty$. First, from Lemma 2.5, we easily see that θ is a Leray-Hopf solution to the (MQG) equation, that is

$$\theta \in L^\infty_{T^*}L^2 \cap L^2_{T^*}\dot{H}^{\frac{\alpha}{2}}.$$

Hence, from the papers [4] and [6] we deduce that for all t_0 in $]0,T^*[$

$$\sup_{t_0\leq t<T^*}\|\theta(t)\|_{C^2_B(\mathbb{R}^2)}<\infty.$$

Therefore, the last assertion of Lemma 2.2 implies

$$M_{t_0}\equiv \sup_{t_0\leq t<T^*}\|u(t)\|_{C^2_B(\mathbb{R}^2)}<\infty.$$

Fix t_0 in $]0,T^*[$ and let $\beta\in\mathbb{N}^2$ any multi-index with $|\beta|\leq 2$. The function $D^\beta\theta$ satisfies

$$\partial_t D^\beta\theta + \Lambda^\alpha D^\beta\theta + u\vec{\nabla}D^\beta\theta = \left[u,D^\beta\right]\vec{\nabla}\theta.$$

Thus, Lemma 2.5 implies that for all t in $[t_0,T^*[$

$$\begin{aligned}\frac{d}{dt}\left\|D^\beta\theta(t)\right\|_2^2 &\lesssim \left\|\left[u,D^\beta\right]\vec{\nabla}\theta(t)\right\|_2\left\|D^\beta\theta(t)\right\|_2\\ &\lesssim M_{t_0}\|\theta(t)\|^2_{H^2}.\end{aligned}$$

Summing on β, we get for all t in $[t_0,T^*[$

$$\frac{d}{dt}\|\theta(t)\|^2_{H^2}\lesssim M_{t_0}\|\theta(t)\|^2_{H^2}.$$

Thanks to The Gronwall inequality, this inequation implies that

$$\sup_{t_0 \leq t < T^*} \|\theta(t)\|_{H^2} < \infty$$

which contradicts Lemma 4.1, since $H^2 \hookrightarrow \dot{H}^{1+\frac{\alpha}{2}}$. We then conclude that $T^* = \infty$.

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